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ANALYSIS ON EVOLUTION PATTERN OF PERIODICALLY DISTRIBUTED DEFECTS

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Abstract—A similar pattern is formed in various materials, when periodically distributed defects evolve. Mathematically, this pattern formation is understood as the consequence of symmetry breaking, while physically it is caused by interaction effect which vary depending on materials or defects. In examining the nature of the interaction effects, this paper analyzes the bifurcation induced growth of a periodic array of defects. With the aid of group-theoretic bifurcation analysis, it is clearly shown that when the uniform pattern (the evolution of all defects) is broken, only the alternate pattern (the evolution of every second defect) can take place for smaller defects, as often observed in nature. Therefore, two defects should be considered to examine a possible bifurcation of periodic defects. Furthermore, the conclusion obtained can be extended to explain the phenomena whereby every second, fourth, and then eighth defect continue to evolve, and whereby alternate bifurcation is repeated successively until the evolution is localized. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Pattern formation is a subject which has been studied in various fields of science; Taylor (1923) was probably the first researcher to make a systematic analysis of a pattern appearing in a viscous fluid (see also Koschmieder (1966)). A number of mathematicians have attempted to establish a unified principle for pattern formation, applying group theory to bifurcation analysis; see, for instance, Sattinger (1979), Golubitsky and Shaeffer (1985) and Golubitsky *et al.* (1988). In the field of applied solid mechanics, a set of defects such as cracks, voids or dislocations evolve forming a certain periodic pattern (see, e.g., Nemat-Nasser *et al.* (1978), Pollard *et al.* (1982), Archambault *et al.* (1993), Ikeda *et al.* (1994)). A typical example is the pattern formation in cracking. When geomaterials are subjected to shearing, a regularly arranged array of cracks develops. Crack size ranges from the order of kilometers (earthquake faults appearing on a ground surface), to the order of centimeters (Riedel shear in a sand specimen); see Hori and Vaikuntan (1996) for a concise list of references. A pattern of similar type is observed for other brittle materials (Ikeda *et al.* (1993)).

To observe the evolution pattern of Riedel shear, Oguni *et al.* (1996) have developed a new experimental method.† In this, the process of Riedel shear formation is monitored for a transparent gelatin sample, and this process is captured with a CCD camera; see

† The standard experiment for Riedel shear uses a plate specimen and applies shear forces at the plate edges. Conditions on the edges may influence the pattern of Riedel shear. The new method applied torsional shearing to a circular plate specimen, and the effects of edge conditions on the pattern formation are removed. The cracks can be controlled so that they grow in a stable manner by applying a suitable confining pressure.

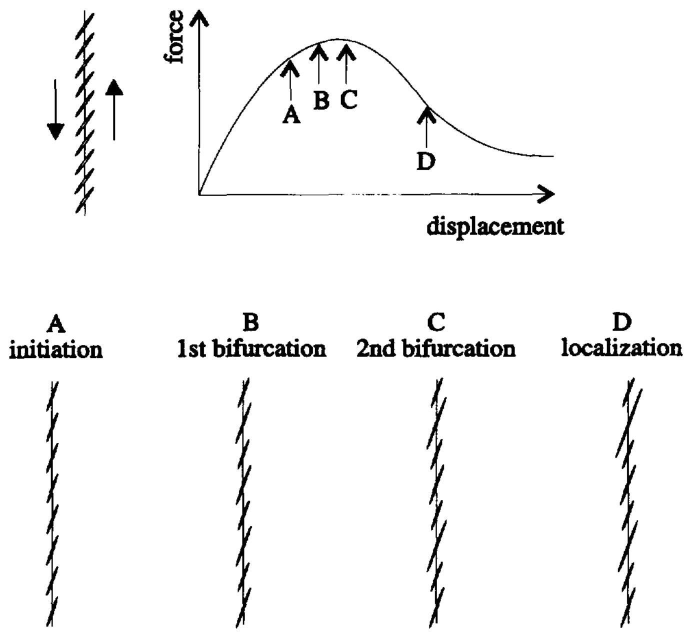

Fig. 1. Crack growth pattern observed in torsion shear experiment.

Appendix A. As illustrated in Fig. 1, this reveals the following four stages for the Riedel shear formation: (A) microcracks of the same size and orientation are initiated at an equal distance from each other, and grow equally as the load increases; (B) at a certain load, every second crack grows equally while the others stop; (C) then every second crack among these growing ones continues to grow as the load further increases; (D) some of the growing cracks start to propagate rapidly and extend to the surface. A similar process of successive bifurcation is observed for other materials subjected to thermal loading; Karihaloo and Nemat–Nasser (1980) studied the initiation and growth of a periodic array of cracks which appear in a glass plate when it is suddenly cooled (see also Keer *et al.* (1993)); and it is widely accepted that equally spaced thermal cracks are initiated on the surface of a cylindrical concrete specimen when it is subjected to higher temperatures.

A periodic structure is used as a mechanical model for such regularly distributed defects. While a cell which contains one defect is analyzed when all defects evolve identically, it is necessary to examine a cell with plural defects for a bifurcated evolution. Since there is no limitation on the number of defects to be examined in such a bifurcation analysis, one needs the minimum number of defects. Many observations report that every second defect evolves after a certain amount of evolution of all defects (Vaikuntan and Hori (1996)); the Riedel shear illustrated in Fig. 1 and the thermally induced cracks (Karihaloo and Nemat–Nasser (1980)) are a typical example. Based on these observations, examination of two defects is sufficient. For a rational bifurcation analysis, therefore, one needs to verify that the alternative pattern is the one that appears after the uniform pattern is broken. Since the alternative pattern is observed in various materials, there may be some common nature of interaction effects which vary depending on materials or defects.

The primary objective of this paper is to study the alternative pattern formation that appears when periodically distributed defects evolve. As an illustrative example, we consider an array of two-dimensional slit cracks in a linearly elastic plate. However, we formulate the problem in a manner such that the formulation can be easily extended to a more complicated setting, and make a general group-theoretic bifurcation analysis which can account for the periodicity and the common nature of interaction effects. This differentiates our approach from analyses of specific problems such as thermally induced periodic cracks (Nemat–Nasser *et al.* (1978), Keer *et al.* (1979))† or group-theoretic bifurcation analyses of

† They studied an array of periodic edge cracks which are thermally induced on semi-infinite plates. The stability of the growth of two or three cracks is carefully examined by computing the interaction effects numerically.

governing equations† with assumed *symmetry* (Ikeda *et al.* (1994)). The content of this paper is as follows: Section 2 considers a unit cell of plural cracks for the periodic structure. The number of cracks in the cell is set to be arbitrary. In Section 3, we obtain a formal expression for a governing equation of the crack growth in this unit cell. This expression takes account of interaction effects, which are evaluated by using the equivalent inclusion method. Section 4 carries out a group-theoretic bifurcation analysis on the governing equation. It is shown that for a unit cell with defects of arbitrary number, alternate growth takes place if the uniform growth pattern is broken. Therefore, it is sufficient to consider two defects for a possible bifurcation in the evolution of periodic defects.

2. PROBLEM SETTING

Two kinds of physical equations are considered in analyzing the evolution of defects. The first is for deformation, which is expressed as a boundary problem for a displacement. The second is an evolution law of defects, which gives a condition for quantities such as strain or stress to make a defect evolve. Once the geometry of the defect is prescribed, these quantities are determined by solving the boundary-value problem. Therefore, the evolution law can be expressed in terms of parameters of the defect geometry, and regarded as a governing equation for these parameters.

It is impossible to express the governing equation explicitly if the boundary-value problem‡ does not have a closed-form solution. Examining the nature of the boundary-value problem, however, we may determine a form of the governing equation even though a closed form expression is not obtained. Such a formal expression is useful when the pattern of evolving defects is examined from bifurcated solutions of the governing equation, and a form of the bifurcated solutions can be specified by applying the group theory to exploit the symmetry for the formal expression (Goubitsky *et al.* (1985), Pollard *et al.* (1982), Ikeda *et al.* (1994)). This analysis does not account for the mechanism whereby interaction effects among defects induce the bifurcation, since the effects vary depending on each boundary-value problem. To give another restriction to the formal expression beside the symmetry, we have to identify some common properties of the interaction effects. This can provide a new insight into the form of bifurcated solutions.

Based on the above consideration, we consider a simple example in which the interaction effects can be formally evaluated; as shown later this will be done by applying the *equivalent inclusion method*. The example is a two-dimensional infinite body which contains an array of identical cracks; see Fig. 2. The crack length and distance are initially a and d, respectively, and it is assumed for simplicity that cracks grow straight without kinking or curving. When the body is subjected to far-field stress, all the cracks grow at the same length to form a uniform growth pattern. As the stress reach σ^o, this uniform pattern of crack growth is broken. It should be noted that the present formulation can also be applicable when periodic cracks grow in a curvilinear manner, if geometrical parameters such as a kinking angle or curvature of the crack are included.

Now, we assume that the pattern satisfies an N-periodicity, i.e., every N-th crack grows in the same manner. The periodicity number N is arbitrary; $N = 2$ corresponds to a case when the alternate pattern is formed, and a case when only one crack grows among many can be studied for a larger N. Therefore, a broad class of bifurcation patterns can be examined by choosing a suitable N. The key question for the present problem is as follows:

> What is the periodicity number N when the uniform pattern of crack growth is broken? And what is the resulting bifurcation pattern?

†In the bifurcation theory, a governing equation for a certain parameter is often called an *equilibrium equation for a displacement*, which is different from the equation of equilibrium for the displacement field used in the applied mechanics.

‡When a specific problem is given, a governing equation can be solved numerically, as done by Nemat–Nasser *et al.* (1979).

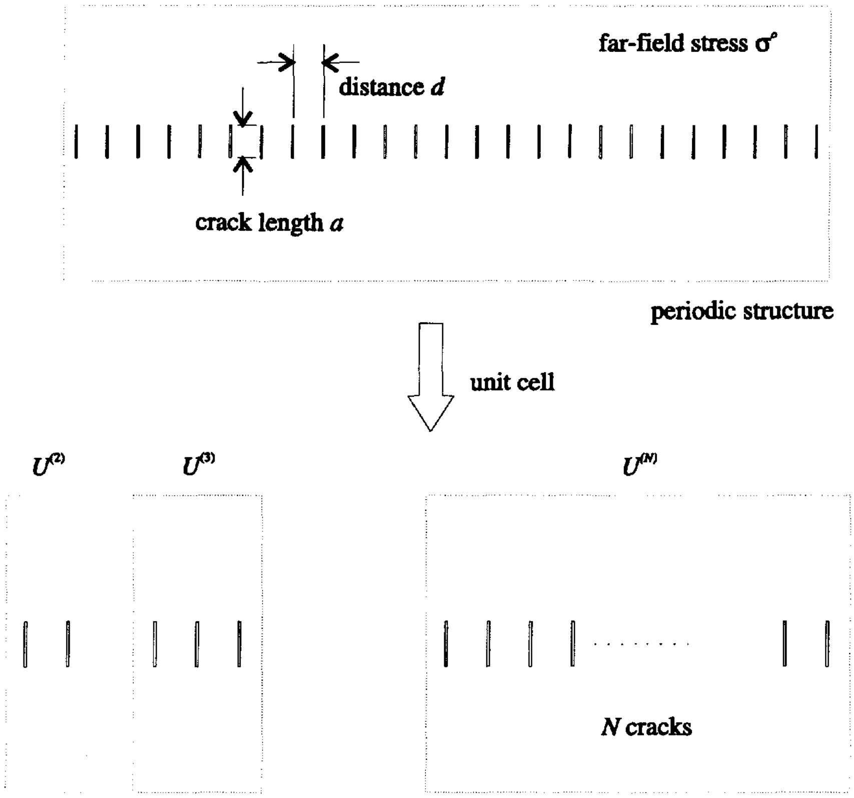

Fig. 2. Periodic structure of cracks.

It should be noted that if the N_1-periodicity and N_2-periodicity need to be examined, it is sufficient to consider $(N_1 \times N_2)$-periodicity alone; the bifurcation of N_1-periodicity or N_2-periodicity is a special case of that of $(N_1 \times N_2)$-periodicity. Without losing generality, therefore, we can assume that N is an even number.

3. FORMULATION

A unit cell† of N-periodicity, $U^{(N)}$, is an infinite strip of width Nd which contains N cracks. The i-th crack is denoted by Ω_i, and its length is a_i ($a_1 = a_2 = \cdots = a_N$). These a_i's are parameters for the crack geometry. For simplicity, denoting by μ the surface energy of the material, we assume the following functional‡ for a displacement field $\mathbf{u}$ and $\{a_i\}$:

$$\Pi(\mathbf{u}, \{a_i\}) = U(\mathbf{u}, \{a_i\}) - W(\mathbf{u}, \sigma^o; \{a_i\}) + \sum_i 2\mu a_i, \tag{1}$$

where U and W are the total strain energy and the external work, respectively, which vary depending on the material properties and loading conditions. A condition of $\delta\Pi = 0$ for fixed $\{a_i\}$ yields a boundary-value problem which determines $\mathbf{u}$, and $\partial\Pi/\partial a_i = 0$ with this $\mathbf{u}$ is§ the fracture criteria for Ω_i. Denoting $\partial\Pi/\partial a_i$ by G_i, we can obtain a governing equation

†A unit cell usually means the minimum unit of a periodic structure; in this problem, for instance, a standard unit cell is a split which contains one crack. Here, however, a set of N these splits, $U^{(N)}$, is called a unit cell, in the sense that the periodic structure is constructed by putting them in a repeated manner.

‡See Ngunen's formulation (Bui (1993)) of a general problem for standard materials.

§$\partial(U - W)/\partial a_i$ is the energy release rate for Ω_i.

of an incremental form for $\{a_i\}$, as

$$\delta G_i = \frac{\partial G_i}{\partial \sigma^o}\delta\sigma^o + \sum_{j=1}^{N} \frac{\partial G_i}{\partial a_j}\delta a_j = 0. \tag{2}$$

The periodicity of $U^{(N)}$ leads to the following: (1) the value of $\partial G_i/\partial\sigma^o$ does not depend on i; and (2) $\partial G_i/\partial a_j$ takes on the same value if the distance of the two cracks, $\min\{|i-j|, N-|i-j|\}d$, is the same. Hence we can rewrite eqn (2) as

$$J_{ij}^{(N)}\,\delta a_j = J_0^{(N)}\,\delta\sigma^o \tag{3}$$

for $i = 1, 2, \ldots, N$, where $J_{ij}^{(N)} = \partial G_i/\partial a_j$ and $J_0^{(N)} = -\partial G_i/\partial\sigma^o$. These $J_{ij}^{(N)}$'s can be expressed as

$$J_{ij}^{(N)} = J_k^{(N)} \quad (k = \min\{|i-j|, N-|i-j|\}). \tag{4}$$

This relation represents the symmetry of $U^{(N)}$. It should be noted that superscript (N) emphasizes that the indicated quantities are for $U^{(N)}$.

When we expand $J_k^{(N)}$ of $k \neq 0$ with respect to kd, the leading term is of the order of $(1/kd)^2$, i.e.,

$$J_k^{(N)} = \sum_{m=2}^{\infty} R_m^{(N)}\left(\frac{1}{kd}\right)^m, \tag{5}$$

where $R_m^{(N)}$'s are (positive) constants. The series on the right side converges if cracks are far from each other and sufficiently small, i.e., $a/d \ll 1$.

The validity of eqn (5) is demonstrated by considering the physical meaning of $J_k^{(N)}$. Since $G_{k+1} = \partial\Pi/\partial a_{k+1}$ gives the difference of the energy release rate of Ω_{k+1} from the surface energy, $J_l^{(N)} = \partial G_{k+1}/\partial a_1$ is the change in Ω_{k+1}'s energy release rate caused by Ω_1's growth. This is the result of the interaction of Ω_1 to Ω_{k+1}. As the energy release rate is a quantity which is computed from strain near the crack tip, we evaluate Ω_1's effects on strain at Ω_{k+1}'s tip. Although the sheer evaluation is almost impossible, we can show that this interaction decays at $O((kd)^{-2})$ as kd goes to infinity, making use of the equivalent inclusion method.† This method replaces Ω_1 with a suitable eigenstress field, $\boldsymbol{\sigma}^*(\mathbf{x})$, which is determined from the geometry and material properties of Ω_1. Whatever the distribution of $\boldsymbol{\sigma}^*(\mathbf{x})$ is, the strain at a point $\mathbf{x}$ near Ω_{k+1}'s edge caused by the presence of Ω_1 can be expressed in terms of Green's function of $U^{(N)}$, $\mathbf{G}^o(\mathbf{x},\mathbf{y})$, as

$$\operatorname{sym}\left\{\nabla\mathbf{x}\otimes\left(\int_{\Omega_1}\mathbf{G}^o(\mathbf{x},\mathbf{y})\cdot\nabla\mathbf{y}\cdot\boldsymbol{\sigma}^*(\mathbf{y})\,\mathrm{d}V_y\right)\right\}$$
$$= -\operatorname{sym}\left\{\int_{\Omega_1}(\nabla\mathbf{x}\otimes(\nabla\mathbf{y}\otimes\mathbf{G}^o(\mathbf{x},\mathbf{y}))):\boldsymbol{\sigma}^*(\mathbf{y})\,\mathrm{d}V_y\right\}, \tag{6}$$

where sym stands for the symmetric part of a second-order tensor in the curvy bracket. If there are several cracks, the resulting strain field is given by taking the sum of the right side

†The equivalent inclusion method (Eshelby (1957)) is an analysis method of a heterogeneous body. The essence of this method is that for an arbitrary heterogeneous body V of elasticity $\mathbf{C}(\mathbf{x})$, a homogeneous body V^o of elasticity $\mathbf{C}^o$ can have the same fields as in V if a suitable *eigenstress* is prescribed in V^o. The eigenstress is given as $\boldsymbol{\sigma}^* = (\mathbf{C}-\mathbf{C}^o):\boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon}$ is the strain in V. The displacement field of V is then expressed in terms of the Green function of V^o, $\mathbf{G}^o(\mathbf{x},\mathbf{y})$, as

$$\mathbf{u}(\mathbf{x}) = \int\mathbf{G}^o(\mathbf{x},\mathbf{y})\cdot(\nabla\mathbf{y}\cdot\boldsymbol{\sigma}^*(\mathbf{y}))\,\mathrm{d}V_y = -\int\nabla\mathbf{y}\times\mathbf{G}^o(\mathbf{x},\mathbf{y}):\boldsymbol{\sigma}^*(\mathbf{y})\,\mathrm{d}V_y;$$

see Nemat-Nasser and Hori (1991) for a more detailed explanation.

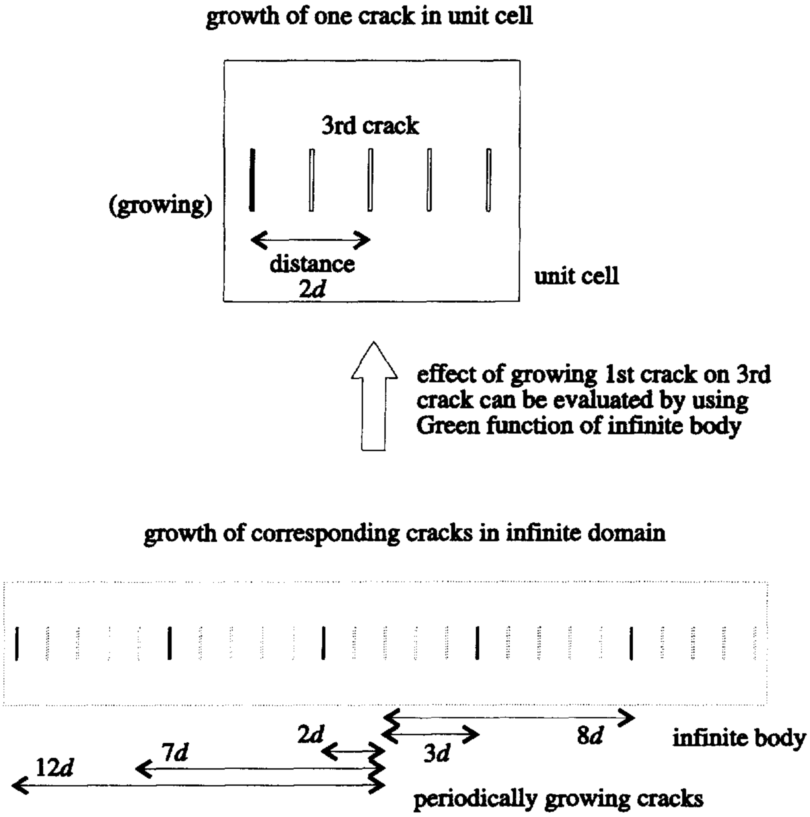

Fig. 3. Evaluation of interaction in unit cell using periodic structure.

of eqn (6) for the eigenstress that corresponds to each crack. As shown in Fig. 3, therefore, the strain field in $U^{(N)}$ is the sum of the strain that is caused by periodically distributed cracks in the infinite body. According to the mean value theory, the right side of eqn (6) is evaluated as

$$-\operatorname{sym}\{(\nabla\mathbf{x}\otimes(\nabla\mathbf{y}\otimes\mathbf{G}^{o}(\mathbf{x},\mathbf{y}^{o})):(\Omega_1\boldsymbol{\sigma}^*(\mathbf{y}^{o})))\},$$

at a point $\mathbf{y}^{o}$ within Ω. Since $\boldsymbol{\sigma}^*(\mathbf{y}^{o})$ approaches a constant as d increases (Eshelby (1957)), the order of the strain due to the presence of interacting cracks is the same as that of the second-order derivative of $\mathbf{G}^{o}(\mathbf{x},\mathbf{y})$. In the two-dimensional† setting, we can evaluate $|\mathbf{G}^{o}(\mathbf{x},\mathbf{y})| \sim \log|\mathbf{x}-\mathbf{y}|$ for large $|\mathbf{x}-\mathbf{y}|$. Therefore, the interaction decays as $O(|\mathbf{x}-\mathbf{y}|^{-2})$.

For more complicated problem settings, such as a doubly periodic case, a governing equation for the crack growth remains of essentially the same form as eqn (3), and its coefficients satisfy eqns (4, 5). This applies even when a more realistic evolution law, non-linear material properties or a dynamic state are considered. It should be emphasized here again that the expansion of the governing equation similar to eqn (5) is admissible when the defects are sufficiently small and far from each other. This is due to the nature of the interaction effects that decay as the inverse of the square of the distance.

4. ANALYSIS ON EVOLUTION PATTERN

Taking advantage of eqns (4, 5) we examine bifurcation solutions of the govern eqn (3) to answer the question posed in Section 2. First, we seek eigenvalues and eigenvectors of an N-by-N Jacobian matrix $[J^{(N)}]$, whose coefficients are $J_{ij}^{(N)}$ with the use of the group-theoretic bifurcation theory; (Murota and Ikeda (1991)). Then, identifying the minimum

†In the three-dimensional setting, we can evaluate $O(|\mathbf{G}^{o}(\mathbf{x},\mathbf{y})|) = O(|\mathbf{x}-\mathbf{y}|^{-1})$ for $|\mathbf{x}-\mathbf{y}| \to \infty$, and hence the interaction decays as the inverse of the cubic of the distance.

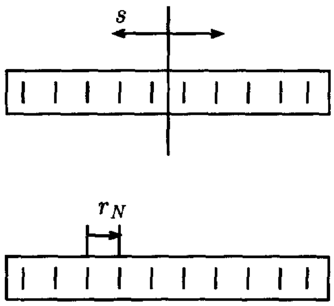

Fig. 4. Transformation by elements s and r_N of D_N.

eigen-value, we consider the bifurcated evolution pattern that corresponds to the associated eigen-vector.

4.1. *Block-diagonalization using group theory*

The unit cell $U^{(N)}$ is invariant under the action of the reflection s with respect to the center line of the cell, as shown in Fig. 4. This $U^{(N)}$ is also invariant under the action of the shift r at a length of d. Such symmetry is labeled by the dihedral group of the degree N

$$D_N = \{r^k, sr^k \,|\, k = 0, 1, \ldots, N-1\} \tag{7}$$

with $r^N = s^2 = (sr)^2 = e$. Here e denotes the unit transformation and the parentheses $\{\cdot\}$ indicate that the relevant group is made up of the elements therein. Then it is easy to see that the incremental governing eqn (2) is *equivariant* with respect to D_N, that is,

$$T_{ij}(g)\, \delta G_j(\sigma^o, \{a_k\}) = \delta G_i(\sigma^o, T_{kj}(g)\{a_j\}), \tag{8}$$

for g in D_N. Here, $T_{ij}(g)$ is a coefficient of N-by-N representation matrix $[T(g)]$; $[T(r)]$ and $[T(s)]$ are

$$[T(r)] = \begin{bmatrix} & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & 1 \\ 1 & & & & \end{bmatrix}, \quad [T(s)] = \begin{bmatrix} & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & \\ \cdot & & & & \\ 1 & & & & \end{bmatrix},$$

and $T(r^k) = [T(r)]^k$ and $T(sr^k) = [T(s)][T(r)]^k$. It should be understood that eqn (8) is *naturally* satisfied for any governing equation with N-periodicity.

Taking advantage of eqn (8), we can easily find eigen-values and eigen-vectors of a matrix $[J^{(N)}]$ whose coefficients are $J_{ij}^{(N)}$'s. Murota and Ikeda (1991) showed that a Jacobian matrix of a D_N-equivalent system can be put into a block-diagonal form through a suitable transformation matrix which can be determined in compatibility with the set of irreducible representations of D_N; see also Ikeda and Murota (1991) for the block-diagonalization and Healey (1985), Golubitsky (1988) and Dinkevich (1991) for the irreducible representation of a D_N equivariant system. Since $[J^{(N)}]$ is a Jacobian matrix of δG_i, its transformation matrix is given as

$$[H^{(N)}] = [[h_{N+}^{(N)}], [h_{N/2-}^{(N)}], [h_{1+}^{(N)}], \ldots, [h_{N/2-1-}^{(N)}]]. \tag{9}$$

where $[h_{j\pm}^{(N)}]$ are N-by-1 submatrices which are given as

$$[h_{N+}^{(N)}] = \frac{1}{\sqrt{N}}[\cos(2\pi), \cos(2\pi 2), \ldots, \cos(2\pi N)]', \tag{10}$$

$$[h_{N/2-}^{(N)}] = \frac{1}{\sqrt{N}}[\cos(\pi 1), \cos(\pi 2), \ldots, \cos(\pi N)]', \tag{11}$$

and

$$[h_{j+}^{(N)}] = \frac{2}{\sqrt{N}}[\cos(\pi j 1/N), \cos(\pi j 3/N), \ldots, \cos(\pi j(2N-1)/N)]', \tag{12}$$

$$[h_{j-}^{(N)}] = \frac{2}{\sqrt{N}}[\sin(\pi j 1/N), \sin(\pi j 3/N), \ldots, \sin(\pi j(2N-1)/N)]' \tag{13}$$

for $j = 1, 2, \ldots, N/2-1$; see Appendix B for detailed derivations. Using this $[H^{(N)}]$, we can obtain the following transformation of $[J^{(N)}]$:

$$[H^{(N)}]'[J^{(N)}][H^{(N)}] = \mathrm{diag}\,[\lambda_{N+}^{(N)}, \lambda_{N/2-}^{(N)}, \lambda_1^{(N)}, \lambda_1^{(N)}, \ldots, \lambda_{N/2-1}^{(N)}, \lambda_{N/2-1}^{(N)}], \tag{14}$$

where diag [·] denotes a diagonal matrix with diagonal entries therein, and the diagonal entries, the eigen-values of $[J^{(N)}]$, are expressed as

$$\lambda_N^{(N)} = J_0^{(N)} + J_{N/2}^{(N)} + 2\sum_{k=1}^{N/2-1} J_k^{(N)}, \tag{15}$$

$$\lambda_{N/2}^{(N)} = J_0^{(N)} + (-1)^{N/2} J_{N/2}^{(N)} + 2\sum_{k=1}^{N/2-1} (-1)^k J_k^{(N)}, \tag{16}$$

and

$$\lambda_j^{(N)} = J_0^{(N)} + \cos(\pi j) J_{N/2}^{(N)} + 2\sum_{k=1}^{N/2-1} \cos\left(\frac{2\pi j}{N}k\right) \tag{17}$$

for $j = 1, 2, \ldots, N/2-1$. It should be noted that an eigenvalue λ_j corresponds to a submatrix $[h_{j\pm}^{(N)}]$ for $j = 1, 2, \ldots, N/2-1, N$.

Without applying group theory, we can compute eigenvalues and eigenvectors of $[J^{(N)}]$ through manipulating eqns (4, 5). As mentioned in the next subsection, however, block-diagonalization using group theory can be directly applied when a_i's and G_i's are vector-valued quantities and corresponding $J_{ij}^{(N)}$'s are tensors. It is not easy to transform these tensor quantities into matrices and to compute their eigen-values and eigen-vectors.

4.2. *Minimum eigenvalue*

Although the values of $\lambda_j^{(N)}$ is not determined unless $J_k^{(N)}$'s are given, we can prove that $\lambda_{N/2}^{(N)}$ always takes on the minimum among $\lambda_j^{(N)}$'s when $J_k^{(N)}$ satisfies eqn (5). That is,

$$\lambda_j^{(N)} - \lambda_{N/2}^{(N)} = (\cos(\pi j) - (-1)^{N/2}) J_{N/2}^{(N)} + 2\sum_{k=1}^{N/2-1} \left(\cos\left(\frac{2\pi j}{N}k\right) - (-1)^k\right) J_k^{(N)} > 0 \tag{18}$$

for $j = 1, 2, \ldots, N/2-1$ or N. These eigenvalues are positive when cracks grow uniformly, since the governing eqn (2) is derived from the functional which computes the total energy

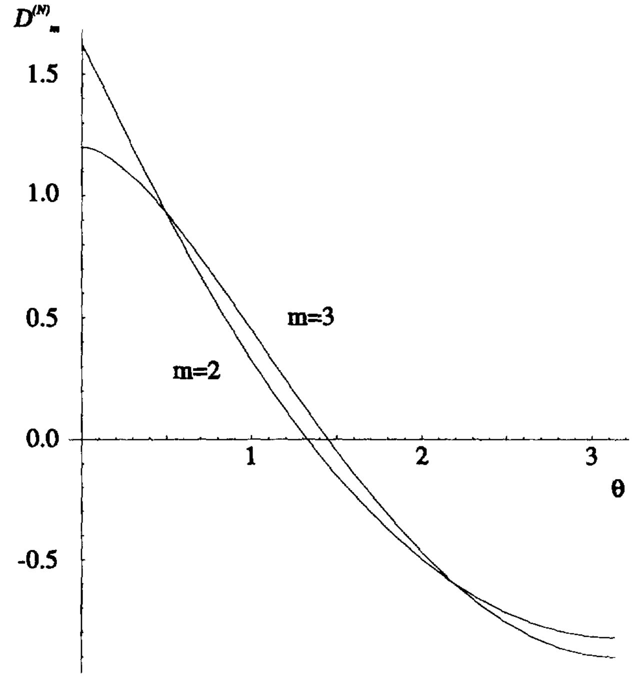

Fig. 5. Function given by $dD_m^{(N)}/d\theta$ as $N \to \infty$.

of the periodic structure; see eqn (1). Hence, it is possible that this eigen-value could be negative while the others remain positive.

Equation (18) is proved as follows: in view of eqn (5), a sufficient condition is

$$\left(\frac{N}{2}\right)^{-m}\left(\cos\left(\frac{2\pi j}{N}\frac{N}{2}\right)-(-1)^{N/2}\right)+2\sum_{k=1}^{N/2-1}k^{-m}\left(\cos\left(\frac{2\pi j}{N}k\right)-(-1)^{k}\right)>0, \tag{19}$$

for $j = 1, 2, \ldots, N/2-1$ and† for $m = 2, 3, \ldots$. The first term on the left side of eqn (19) is non-negative. Defining a function $D_m^{(N)}(\theta)$ as

$$D_m^{(N)}(\theta) = \sum_{k=1}^{N/2-1} k^{-m}\cos(k\theta), \tag{20}$$

we can express the second term as $2(D_m^{(N)}(2\pi j/N)-D_m^{(N)}(\pi))$. If $D_m^{(N)}(\theta)$ decreases monotonically with respect to θ, i.e.,

$$\frac{dD_m^{(N)}}{d\theta}(\theta) = -\sum_{k=1}^{N/2-1} k^{-m+1}\sin(k\theta) < 0, \tag{21}$$

in $0 < \theta < \pi$ for $m = 2, 3, \ldots$, then eqn (19) holds. Since $dD_m^{(N)}/d\theta$ becomes 0 at $\theta = 0$ and π for any m, $dD_m^{(N)}/d\theta$ is negative in $0 < \theta < \pi$ if $d^2D_m^{(N)}/d\theta^2$ increases monotonically. From $d^2D_m^{(N)}/d\theta^2 = -D_{m-2}^{(N)}$, it follows that eqn (21) holds for any m, if it holds for $m = 2$ and 3. The case of $m = 2$ and 3 are shown in Appendix C, and eqn (18) is proved. It should be noted that $dD_m^{(N)}/d\theta$ is a truncated Fourier series expansion of a function. Figure 5 illustrates function for $m = 2$ and $m = 3$ which are numerically computed. These functions appear monotonically decreasing in $0 < \theta < \pi$, and it is seen that eqn (21) holds when N is sufficiently large.

†The case of $j = N$ is trivial.

By definition, $\lambda_j^{(N)}$'s are positive when cracks grow steadily. At the critical σ^o when the uniform growth is broken, $\lambda_j^{(N)}$'s satisfy

$$\lambda_j^{(N)} > \lambda_{N/2}^{(N)} \geqslant 0 \tag{22}$$

for $j \neq N/2$. The resulting bifurcated solution of δa_i's must correspond to the eigenvector $[h_{N/2}^{(N)}]$. If $\lambda_{N/2}^{(N)}$ becomes negative for an infinitesimal growth, the most unstable solution will be of the form of

$$[\delta a] = \delta a[1, \quad 0, \quad 1, \quad 0, \ldots, 0]', \tag{23}$$

since the form of the trivial solution is $[h_{N+}^{(N)}]$ and that of the bifurcated solution is $[h_{N/2-}^{(N)}]$; see eqns (10), (11). This alternate pattern is the bifurcation solution that first becomes unstable for any even N. Although $U^{(N)}$ with an odd N cannot have a bifurcated solution given by eqn (23), the bifurcation of $U^{(N)}$ is a special case of that of $U^{(2N)}$, and hence the alternate pattern appears earlier than other patterns. Therefore, we can conclude that

> Provided that eqn (5) holds, an alternate pattern of crack growth appears when the uniform growth pattern is broken. Therefore, it is sufficient to consider a unit cell of two cracks to analyze possible bifurcation of the crack growth.

Although there are some conditions required, the conclusion drawn from the present analysis is general, and can be applied to pattern formation in various types of periodically distributed defects. For instance, the conclusion can be applied to explain a subsequent alternate bifurcation, i.e., every second crack of those second cracks which have continued growing will itself grow as the stress further increases. This bifurcation process can be repeated until this crack becomes large and expansion similar to eqn (4) no longer holds. The conclusion is also applicable to an array of three-dimensional cracks which grow in an arbitrary fashion, or to a doubly- or triply-periodic structure of arbitrary defects; see Appendix D for generalization of this conclusion.

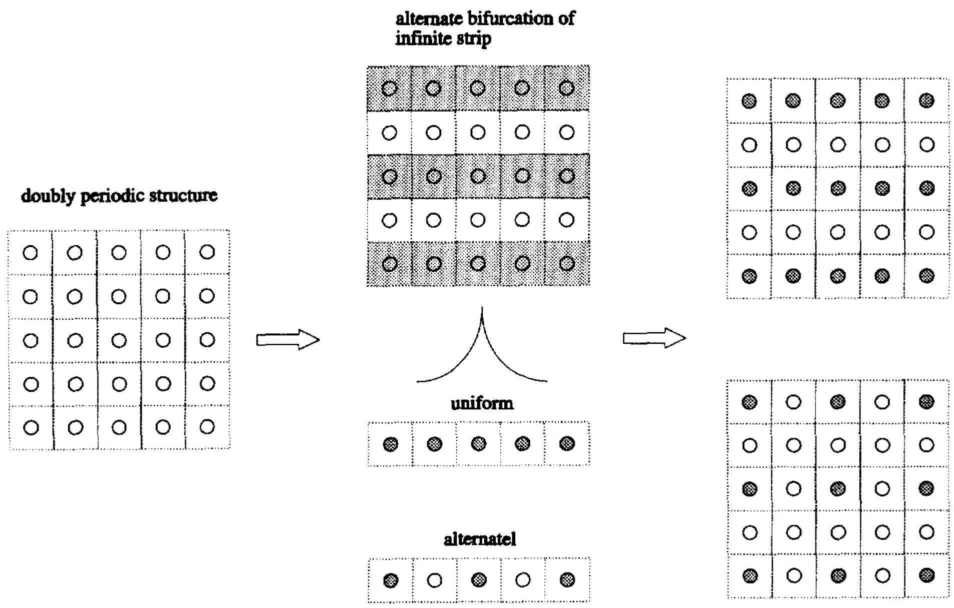

Fig. 6. Doubly-periodic array of defects.

5. CONCLUDING REMARKS

Examining the nature of the interaction effects, we determine a form of the governing equation for the evolution of periodically distributed defects. The group-theoretic bifurcation analysis of this governing equation shows that when the defects are relatively small, defect evolution always leads to an alternate pattern. Therefore, in investigating the possible bifurcation of evolving periodic defects, analysis of a unit cell with only two defects is sufficient. Furthermore, since no special assumptions are made in drawing this conclusion, it can be extended to a more complicated setting; for instance, this conclusion is applicable to the explanation of a successive occurrence of alternate pattern formation, which is observed in the Riedel shear experiment.

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APPENDIX A: TORSIONAL SHEAR EXPERIMENT

Several experiment methods have been proposed to generate a periodic array of cracks in geomaterials. Most of the experiments use a long specimen to reduce disturbances caused by the edges of the specimen, though it is not easy to remove them perfectly. The torsional shear experiment proposed is a new technique which puts a circular-disk sample on a circular plate and generates a periodic array of cracks around the edge of the plate, by rotating the plate and causing torsional shearing. Although the sample is initially in an axially symmetric condition, the gradual rotation of the plate induces an array of microcracks of the identical size and equal distance. Since the radius of the plate is much larger than these cracks, they appear aligned in a straight line. Hence, these cracks are regarded as a model of the Riedel shear which appears along a straight edge of sliding plates in conventional experiments. There are no boundary effects in this technique, and the initiation and growth of cracks is almost uniform along the circular edge. In this experiment, a transparent material (gelatin) is used. The whole process is

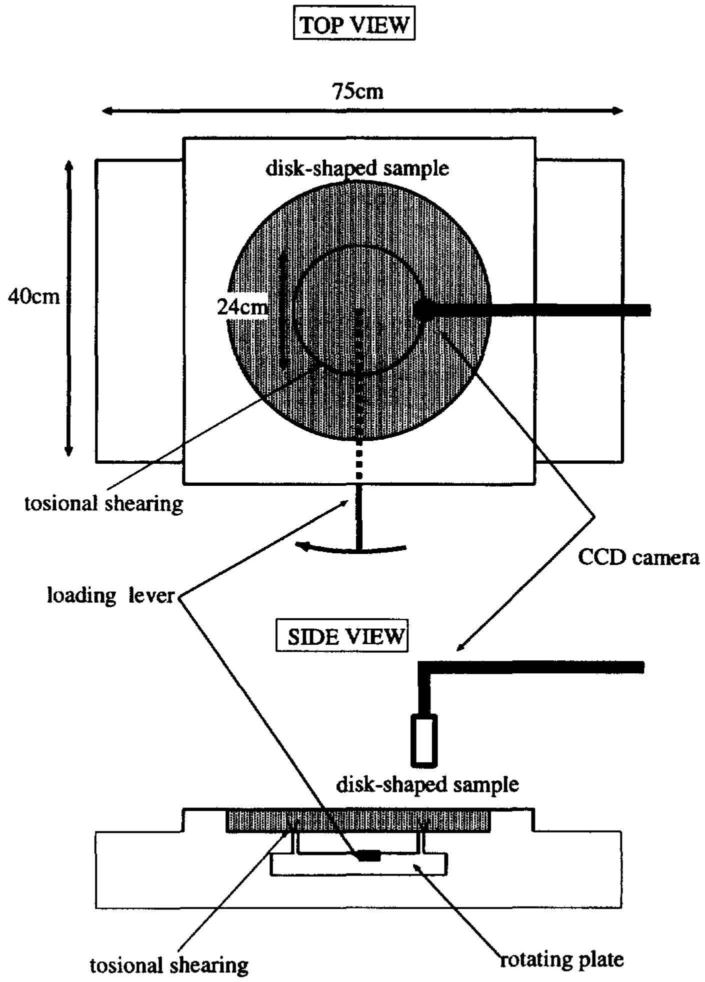

Fig. A1. Equipment for torsion experiment.

monitored through a CCD camera, and the images are recorded. Figure A1 shows schematic top and side views of the experiment equipment.

APPENDIX B: IRREDUCIBLE REPRESENTATION OF D_N

To analyze the bifurcation of a D_N-equivariant system, we consider the irreducible representations of the group D_N (Healey (1985), Golubitsky *et al.* (1988), Muroa and Ikeda (1991), Dinkevich (1991), Ikeda and Murota (1991)). This D_N has four one-dimensional irreducible representations, $N\pm$ and $N/2\pm$, and $N/2-1$ two-dimensional ones, $1, 2, \ldots, N/2-1$. The former is characterized by the following 1-by-1 irreducible representation matrices:

$$\begin{aligned}
T_{N+}(r) &= 1, \quad T_{N+}(s) = 1, \\
T_{N-}(r) &= 1, \quad T_{N-}(s) = -1, \\
T_{N/2+}(r) &= -1, \quad T_{N/2+}(s) = 1, \\
T_{N/2-}(r) &= -1, \quad T_{N/2-}(s) = -1,
\end{aligned}$$

and the latter by the following 2-by-2 ones:

$$[T_j(r)] = \begin{pmatrix} \cos(2\pi j/N) & -\sin(2\pi j/N) \\ \sin(2\pi j/N) & \cos(2\pi j/N) \end{pmatrix}, \quad [T_j(s)] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

see, for instance, Murota and Ikeda (1991) for detailed explanation. Using these irreducible representations, we can determine a submatrix of the transformation matrix $[H^{(N)}]$ given by eqn (9). The submatrix satisfies the following relationship:

$$[T_\mu(g)][h_\mu^{(N)}]' = [h_\mu^{(N)}][T_\mu(g)],$$

for g in D_N and $\mu = N\pm,\ N/2\pm,\ 1,\ldots,N/2-1$. This $[h_{j\pm}^{(N)}]$ is decomposed from a N-by-2 matrix $[h_j^{(N)}] = [[h_{j+}^{(N)}][h_{j-}^{(N)}]]$ which satisfies $[T_j(g)][H_j^{(N)}]' = [H_j^{(N)}][T_j(g)]$, for $j = 1, 2, \ldots, N/2-1$. It should be noted that the submatrices $[h_{N-}^{(N)}]$ and $[h_{N/2+}^{(N)}]$ do not exist in eqn (9), since the present D_N is a particular case that corresponds to the Case 1ZM in Murota and Ikeda (1991).

APPENDIX C: PROOF OF $\mathrm{d}D_m^{(N)}/\mathrm{d}\theta < 0$ for $m = 2$ and 3

First, we consider the case of $m = 2$. The second derivative of $D_m^{(N)}$ defined by eqn (20) becomes

$$\frac{\mathrm{d}^2 D_2^{(N)}}{\mathrm{d}\theta^2} = -\frac{1}{2}\left(\frac{\sin\dfrac{(N-1)\theta}{2}}{\sin\dfrac{\theta}{2}} - 1\right). \tag{C1}$$

Since $\sin(\theta/2)$ increases monotonically in $0 < \theta < \pi$, $D_2^{(N)}$ takes on the local maximum at $\theta = 4(k+1)\pi/(N-1)$ in $4k\pi/(N-1) < \theta \leqslant 4(k+1)\pi/(N-1)$. Dividing this region into $4k\pi/(N-1) < \theta < (4k+2)\pi/(N-1)$ and $(4k+2)\pi/(N-1) < \theta < 4(k+1)\pi/(N-1)$, we evaluate

$$\int_{4k\pi/(N-1)}^{4(k+1)\pi/(N-1)} \frac{\sin\dfrac{(N-1)\theta}{2}}{\sin\dfrac{\theta}{2}}\,\mathrm{d}\theta > \frac{4}{N-1}\left(\frac{1}{\sin\dfrac{(2k+1)\pi}{N-1}} - \frac{1}{\sin\dfrac{2(k+1)\pi}{N-1}}\right). \tag{C2}$$

Taking the sum of eqn (C2) for $k = 0$ to $k = l$, we obtain the following inequality for $\mathrm{d}^2 D_2^{(N)}/\mathrm{d}\theta^2$.

$$2\int_0^{4(l+1)\pi/(N-1)} \frac{\mathrm{d}^2 D_2^{(N)}}{\mathrm{d}\theta^2}\,\mathrm{d}\theta \leqslant -\frac{(N-6-7l)\pi}{N-1} + \frac{4}{\pi}\,\frac{\dfrac{\pi}{N-1}}{\sin\dfrac{\pi}{N-1}}. \tag{C3}$$

If N is sufficiently large, the right side of eqn (C3) becomes negative for l satisfying $4(l+1)/(N-1) \leqslant 1/4$. Therefore, $\mathrm{d}^2 D_2^{(N)}/\mathrm{d}\theta^2 < 0$, and this implies $\mathrm{d}D_2^{(N)}/\mathrm{d}\theta < 0$ in $0 < \theta < \pi/4$.

In a similar manner, we can prove $\mathrm{d}D_2^{(N)}/\mathrm{d}\theta < 0$ in $\pi/4 < \theta < \pi$. To make expressions simpler, we replace θ by $\phi = \pi - \theta$ and $D_2^{(N)}$ by $E_2^{(N)}(\phi) = D_2^{(N)}(\pi-\phi)$, and show $\mathrm{d}E_2^{(N)}/\mathrm{d}\phi > 0$. The second derivative of $E_2^{(N)}$ is

$$\frac{\mathrm{d}^2 E_2^{(N)}}{\mathrm{d}\phi^2} = \frac{1}{2}\left(1 + \frac{\cos\dfrac{(N-1)\phi}{2}}{\cos\dfrac{\phi}{2}}\right). \tag{C4}$$

Since $\cos((N-1)\phi/2)/\cos(\phi/2)$ takes on the local minimum at $\phi = (4k+3)\pi/(N-1)$ in $(4k-1)\pi/(N-1) < \phi < (4k+1)\pi(N-1)$, its integration from $(4k-1)\pi/(N-1)$ to $(4k+3)\pi/(N-1)$ can be evaluated like eqn (C2). Hence, we obtain

$$\int_{(4k-1)\pi/(N-1)}^{(4k+3)\pi/(N-1)} \frac{\mathrm{d}^2 E_2^{(N)}}{\mathrm{d}\phi^2}\,\mathrm{d}\phi \geqslant \frac{2}{N-1}\left(\frac{1}{\cos\dfrac{(4k-1)\pi}{2(N-1)}} - \frac{1}{\cos\dfrac{(4k+3)\pi}{2(N-1)}} + \frac{\pi}{2} + 1\right). \tag{C5}$$

This inequality corresponds to eqn (C2). Therefore, we can show $\mathrm{d}E_2^{(N)}/\mathrm{d}\phi > 0$ in $0 < \phi < 3\pi/4$ for sufficiently large N, using eqn (C5) in the same manner as eqn (C2) for $D_2^{(N)}$.

Next, we consider the case of $m = 3$. This case is easier than the case of $m = 2$, since $\mathrm{d}D_3^{(N)}/\mathrm{d}\theta$ is uniformly convergent as $N \to \infty$. The converging function is $\theta \log 2 + \int_0^\theta \log(\sin(\theta/2))\,\mathrm{d}\theta$, and is negative in $0 < \theta < \pi$. Hence, except for the neighborhood of $\theta = 0$ and $\theta = \pi$, $\mathrm{d}D_3^{(N)}/\mathrm{d}\theta < 0$ when N is large. Evaluating the integrand of $D_3^{(N)}$ and its derivatives in the same manner as for the case of $m = 2$, we can easily show $\mathrm{d}D_3^{(N)}/\mathrm{d}\theta < 0$ in these two regions, and prove that $\mathrm{d}D_3^{(N)}/\mathrm{d}\theta < 0$ in $0 < \theta < \pi$.

APPENDIX D: GENERALIZATION OF PRESENT ANALYSIS

As mentioned for the model experiment in Section 1, the alternate pattern formation can take place successively; for instance, only every fourth crack starts to grow after alternate cracks grow to some extent. We can explain this phenomenon using the present group-theoretic bifurcation analysis. Viewing a pair of growing and non-growing cracks as a cell of a periodic structure, we set a new unit cell of $N/2$-periodicity, $U^{(N/2)}$, which contains $N/2$ pairs. Ω^i now stands for the i-th pair of cracks, and δa_i is a vector for the length of the two cracks of Ω^i. A governing equation for δa_i's becomes essentially the same form as eqn (3), except that δa_i and δG_i become vectors. This $U^{(N)}$ is still $D_{N/2}$-invariant, and coefficients of the Jacobian matrix admit the expansion similar to eqn (5). Therefore, $D_{N/4}$-invariant alternate growth (growth of every fourth crack) appears when the $D_{N/2}$-invariant growth

(growth of every second crack) is broken. Such a process can be repeated M times. While the equation is now for 2^M-th order vectors, the unit cell is $D_{N/2^M}$ invariant. This successive occurrence of the alternate pattern may not happen, if growing cracks become large and the expansion similar to eqn (5) no longer holds.

The results can be applied to a doubly-periodic array of defects with some restriction; see Fig. 6. The symmetry of this periodic structure is as follows: denoting the defect at the i-th row and j-th column by Ω_{ij}, and its length and governing equation by a_{ij} and $G_{ij}(\sigma^o, \{a_{kl}\})$, respectively, we have

$$T_{ijmn}(g)\,\delta G_{mn}(\sigma^o, \{a_{kl}\}) = \delta G_{ij}(\sigma^o, T_{klmn}(g)\{a_{mn}\}), \tag{D1}$$

where g is an invariant action of this periodic structure; see eqn (8). If the doubly periodicity is viewed† as a *combination* of two periodic arrays, then, the periodic structure is an array of infinite strips each of which contains the identical periodic array of defects, as shown in Fig. 6. When the uniform evolution ceases, every two strips has evolving defects. In each strip, the evolving pattern is either uniform or alternate. Similar conclusions can be applied to a three-dimensional structure which satisfies the triply periodicity. Again, if this periodicity is viewed as the combination of three periodic arrays, an alternate pattern, such as the growth of cracks in every two layers, would be formed after the uniform growth is broken. It should be emphasized that in the double or triple periodicity, such an alternate pattern appears only when the interaction effects among defects admits the expansion similar to eqn (5).

†This is a restriction for the double periodicity. Indeed, this restriction corresponds to a case when T_{ijmn} in eqn (D1) is given as the product of two matrices as $T_{ijmn} = T^1_{im}T^2_{jn}$.